

## Drowning Out Rumor: Dynamical Models of the Interaction between Spreaders of and Exposed to Truth and Rumor Spreading

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Social networking sites have become instrumental in spreading information online, which unfortunately includes rumors and misinformation. Past studies have investigated the spread of rumors without considering that truth may spread simultaneously. This study considers both the spread of truth and rumor and investigates the result of their coexistence in a population. We formulated the spreader-spreader interaction model and the exposed-spreader interaction model based on the epidemiological SEIR model. In the spreader-spreader interaction model, the spreaders try to influence the opposing spreaders whereas, in the exposed-spreader interaction model, the spreaders try to influence the exposed individuals from the opposing side. In our study, we calculated the reproduction numbers for truth and rumor, determined the stability of the model at the equilibrium points, and determined an approach for spreading truth while halting rumors. In the spreader-spreader model, we have shown that increasing the removal rate of Spreaders of Rumor decreases the prevalence of rumor in the population. Whereas in the exposed-spreader model, we showed that increasing the conversion rate of Exposed to Rumor to Exposed to Truth significantly increases the amount of Exposed to Truth in the population, thereby making it an effective mechanism for promoting the spread of truth. In conclusion, it is harder to control the information epidemic in the exposed-spreader model but, compared to the spreader-spreader model, information spreads faster according to the exposed-spreader model. For certain information to be endemic to a population, our study proposes that spreaders take advantage during the first few days of the information diffusion process.

Keywords: dynamical systems, information epidemic, ordinary differential equations, rumor spread

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## INTRODUCTION

Social networking sites are online platforms that connect us with our real-life friends in the digital realm. As technology progressed, entities like brands, news outlets, and people have utilized and transformed these social networking sites as platforms to spread influences, promote products, and increase audiences (Kandhway and Kuri 2014, Rodrigues 2016). While this phenomenon has benefited both influence-seekers for increased reach and the masses for serendipitous encounters of new information, this has also brought about negative externalities to our societies. One of which is the deliberate spread and amplification of misinformation and rumors, enabled by previously unknown accounts that are emerging and claiming ascendancy over established information sources (Del Vicario *et al.* 2016). Considering their evident societal impact, it is vital for us to understand how information spreads within these platforms and how to control it.

There have been empirical and interpretative studies that characterize online rumors and how they spread on social media ecosystems like Twitter (Andrews *et al.* 2016, Maddock *et al.* 2015, Starbird 2017, Stewart *et al.* 2017) and Facebook. Complementary to these are mathematical models that have been developed to understand the underlying dynamics behind the spread of information and how we can control it. Mathematical models of infectious diseases have been established as a method to project the likely outcome of epidemics. These are helpful in setting policies and designing interventions to control and prevent further spreading. Since the spreading behavior of infectious diseases behave similarly to how information is spread (Wang *et al.* 2018), similar modeling approaches such as using the compartmental SIR model can be used to study the spread of information (Hu *et al.* 2018, Kandhway and Kuri 2014, Rodrigues 2016).

In social computing, rumoring is the process of communicating relevant topics that do not originate from official sources (*i.e.*, established news outlets, national government) and does not follow formal means of dissemination (*i.e.*, news cycles, health bulletins) (Spiro *et al.* 2012). This results in rumors, which are informal information borne out of anxiety and uncertainty. They are taken as truth without verifying their truthfulness and sources.

Over time, several extensions and variations of the classic SIR model have been developed for rumor spreading. Hu *et al.* (2018) developed a compartmental model that considers different attitudes of people towards rumors that affect its spread – such as hesitation, affirmation, and resistance – which they called the Susceptible-Hesitating-Affected-Resistant (SHAR) model (Hu *et al.* 2018). Their numerical simulations revealed that hesitant agents surprisingly advance the spread of rumors, contrary to what people intuit. There is also work done to model both truth and rumor spread in a population. Liu *et al.* (2018b) developed a compartmental model with two types of information spreaders: Spreaders of Rumor and Spreaders of Truth. Their model accounts for interaction between the spreaders and ignorant individuals, as well as between the two types of spreaders.

Another variant of the SIR model takes into account an *exposed* compartment (Keeling and Rohani 2008, Liu *et al.* 2018a) which represents individuals who have been exposed to a disease but are not yet infectious. Analogously, in information spread, the exposed compartment represents individuals who have received certain information but have yet to share it. In the work of Liu and co-authors, they only have one type of *exposed* compartment (Liu *et al.* 2018a) and introduced rumor *attractiveness* and *fuzziness*. Their results suggest that a rumor with high *attractiveness* and *fuzziness* spreads faster. However, it was not clear how the different types of exposure led to the promotion of rumor spreading.

In this study, we extended the model built by Liu *et al.* (2018b) and added the *exposed* compartments to both truth and rumor. If we look back at the social computing definition of rumoring, the core behavior in this phenomena is the increased interaction among people who have been exposed to improvised information (rumors) and among people who have affirmed these rumors and decided to pass it on (Spiro *et al.* 2012). The sharing behavior is well covered in the literature. However, the interaction among exposed people who are still verifying their new information warrants further investigation. Our approach builds on the SEIR model by developing a system of ordinary differential equations and creating two Spreader and two Exposed compartments, one Spreader for Truth and one for Rumor, and one Exposed to Truth and one Exposed to Rumor.

We investigate the dynamics of truth and rumor spreading, considering the coexistence and interactions between Spreaders of Truth and Rumor. We developed two mathematical models based on the SEIR model that describes spreader-spreader and exposed-spreader interactions. From a macro-level, we observed the general effects of changing the rates of spreading truth on the population as a whole. We compute the reproduction number  $R_0$  and analyzed the models' stabilities at their equilibrium point. Further, we generate simulations on proposed control strategies and decide on the best approach to promote the spread of truth while halting the spread of rumors.

## MODEL DESCRIPTION

### Spreader-Spreader Model

We have an active population  $N$  that is inside the sphere of influence of Spreaders of Truth and Spreaders of Rumor. The population is divided into the Ignorant ( $I$ ), Exposed to Truth ( $E_+$ ), Exposed to Rumor ( $E_-$ ), Spreader of Truth ( $S_+$ ), Spreader of Rumor ( $S_-$ ), and Removed ( $R$ ) as shown in Figure 1. This model focuses on the interaction of the spreader of truth and spreader of rumor and how it affects the spread of truth and rumor.

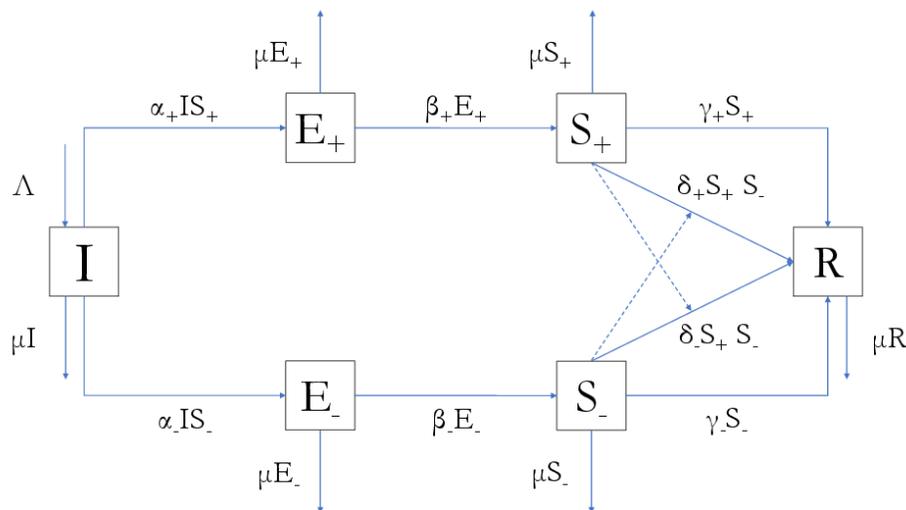


Figure 1. The Spreader-Spreader interaction model.

The inflow  $\Lambda$  in  $I$  represents the entering of Ignorant inside the collective sphere of influence of the spreaders. This could be interpreted as the rate at which the spreaders are able to contact new people. The outflow  $\alpha_+ IS_+$  ( $\alpha_- IS_-$ ) represents the successful dissemination of truth (rumor) to the Ignorant, exposing them to either the truth (rumor). The dotted lines pointing to these flows indicate that  $\alpha_+ IS_+$  and  $\alpha_- IS_-$  is regulated by  $S_+$  and  $S_-$ , respectively. The outflow denoted by  $\mu I$  corresponds to the exiting of Ignorant from the collective sphere of influence of the spreaders.

In this model,  $\alpha_+ IS_+$  is the inflow for  $E_+$  coming from the Ignorant. The transition of individuals from being Exposed to Truth into Spreaders of Truth is denoted by  $\beta_+ E_+$ . Similarly,  $\alpha_- IS_-$  is the transition rate from Ignorant to Exposed  $E_-$ , and  $\beta_- E_-$  is the transition rate from being Exposed to Rumor into Spreader of Rumor.

The removal of the spreaders occurs either due to loose of interest to the topic or conflict of ideas.

When the Spreaders of Truth (rumor) are no longer interested in the topic and withdrew themselves from the circle of discussion, the removal rate of the Spreaders is denoted by  $\gamma_+ S_+$  ( $\gamma_- S_-$ ). This results in removal from the spreading

process. Removal rate from the circle of truth (rumor) spreading due to conflict of ideas is denoted by  $\delta_+S_+S_-$  ( $\delta_-S_-S_+$ ). In this process, illustrated through the broken lines, the spreader of rumor (truth) tries to convince or influence the spreader of truth (rumor).

The Spreader-Spreader model that involves interactions of opposing spreaders is represented by the following system of ODEs:

$$\begin{cases} \dot{I} &= \Lambda - \alpha_+IS_+ - \alpha_-IS_- - \mu I \\ \dot{E}_+ &= \alpha_+IS_+ - \beta_+E_+ - \mu E_+ \\ \dot{E}_- &= \alpha_-IS_- - \beta_-E_- - \mu E_- \\ \dot{S}_+ &= \beta_+E_+ - \gamma_+S_+ - \delta_+S_+S_- - \mu S_+ \\ \dot{S}_- &= \beta_-E_- - \gamma_-S_- - \delta_-S_-S_+ - \mu S_- \\ \dot{R} &= \gamma_+S_+ + \delta_+S_+S_- + \gamma_-S_- + \delta_-S_-S_+ - \mu R \end{cases} \quad (1)$$

where  $\Lambda > 0, \mu > 0, \alpha_+ > 0, \alpha_- > 0, \beta_+ > 0, \beta_- > 0, \gamma_+ > 0, \gamma_- > 0, \delta_+ > 0, \delta_- > 0$  and population  $N = I + E_+ + E_- + S_+ + S_- + R$ . The positivity and boundedness of the equations is shown in Appendix I Proposition 1.

### Exposed-Spreader Model

To investigate the interactions between spreaders and exposed, we built the Exposed-Spreader interaction model shown in Figure 2. In this model, we focus on the dynamics between spreaders and exposed instead of between spreaders.

Instead of having the transitions  $\delta_+S_+S_-$  and  $\delta_-S_-S_+$  from  $S_+$  and  $S_-$ , respectively, into  $R$ , we have the transition of  $\epsilon_+E_+S_-$  from  $E_+$  to  $E_-$  due to the Spreaders of Rumors spreading rumors among the exposed to the truth. At the same time, we have another outflow of  $\epsilon_-E_-S_+$  from  $E_-$  to  $E_+$  due to the Spreaders of Truth spreading truth among the exposed to rumors. The transition rates  $\epsilon_+E_+S_-$  and  $\epsilon_-E_-S_+$  are regulated by their respective Spreaders  $S_-$  and  $S_+$ . The associated diagram of this variant of the mathematical model that involves exposed-spreader interaction is shown in Figure 2.

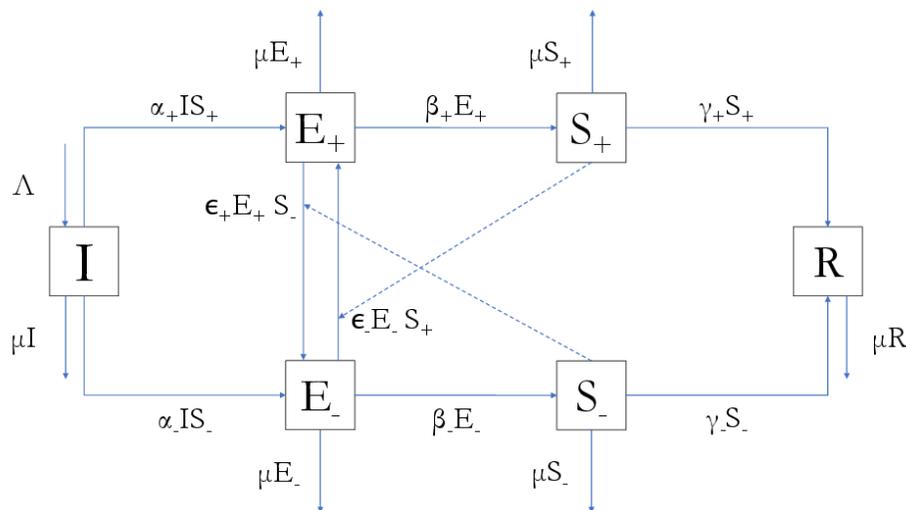


Figure 2. The exposed-spreader interaction model.

$$\begin{cases} \dot{I} &= \Lambda - \alpha_+ IS_+ - \alpha_- IS_- - \mu I \\ \dot{E}_+ &= \alpha_+ IS_+ + \epsilon_- E_- S_+ - \beta_+ E_+ - \epsilon_+ E_+ S_- - \mu E_+ \\ \dot{E}_- &= \alpha_- IS_- + \epsilon_+ E_+ S_- - \beta_- E_- - \epsilon_- E_- S_+ - \mu E_- \\ \dot{S}_+ &= \beta_+ E_+ - \gamma_+ S_+ - \mu S_+ \\ \dot{S}_- &= \beta_- E_- - \gamma_- S_- - \mu S_- \\ \dot{R} &= \gamma_+ S_+ + \gamma_- S_- - \mu R \end{cases} \quad (2)$$

where  $\Lambda > 0$ ,  $\mu > 0$ ,  $\alpha_+ > 0$ ,  $\alpha_- > 0$ ,  $\beta_+ > 0$ ,  $\beta_- > 0$ ,  $\gamma_+ > 0$ ,  $\gamma_- > 0$ ,  $\epsilon_+ > 0$ ,  $\epsilon_- > 0$  and population  $N = I + E_+ + E_- + S_+ + S_- + R$ . The positivity and boundedness of the equations are shown in Appendix I Proposition 2.

## ANALYSIS OF THE MATHEMATICAL MODEL

Given the spreader-spreader interaction and exposed-spreader interaction models, we investigate the presence of four equilibrium points – (1) truth-rumor-free, (2) truth-free, (3) rumor-free, and (4) truth-rumor coexistence. The truth-rumor-free equilibrium point means that neither truth nor rumor exists in the population. A population where truth does not exist is truth-free and, likewise, a population with no rumor is a rumor-free population. When both truth and rumor simultaneously exist, the population has truth-rumor coexistence.

### Analysis of Spreader-Spreader Interaction Model

**Truth-Rumor-free equilibrium and the basic reproduction number.** We can obtain the Truth-and-Rumor-free equilibrium,  $E_{SS}^0$ , by setting the left side of the system to 0. It is easy to see that we obtain  $E_{SS}^0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ . Now, we use the next generation matrix to find the basic reproduction number  $R_{SS}^0$ . In this case,  $R_{SS}^0$  is defined to be the anticipated new generation of truth-rumor spreaders influenced by a member of the spreader population. Since we have two different kinds of spreaders, we set  $R_{SS}^0 = \max\{R_{SS}^{01}, R_{SS}^{02}\}$  where  $R_{SS}^{01}$  and  $R_{SS}^{02}$  pertains to the reproduction number of truth and rumor, respectively.

To solve for  $R_{SS}^{01}$ , we let  $\bar{x} = (E_+, S_+)^T$ . From (1), we get  $\mathcal{F}(\bar{x})$  and  $\mathcal{V}(\bar{x})$  where:

$$\mathcal{F}(\bar{x}) = \begin{pmatrix} \alpha_+ IS_+ \\ 0 \end{pmatrix} \text{ and } \mathcal{V}(\bar{x}) = \begin{pmatrix} \beta_+ E_+ + \mu E_+ \\ -\beta_+ E_+ + \gamma_+ S_+ + \delta_+ S_+ S_- + \mu S_+ \end{pmatrix}. \quad (2)$$

From (3), we obtain  $F = \begin{pmatrix} 0 & \frac{\alpha_+ \Lambda}{\mu} \\ 0 & 0 \end{pmatrix}$  and  $V = \begin{pmatrix} \beta_+ + & 0 \\ -\beta_+ & \gamma_+ + \end{pmatrix}$ .

Therefore, the reproduction number of truth  $R_{SS}^{01}$  is the largest eigenvalue of  $FV^{-1}$  shown below:

$$R_{SS}^{01} = \rho(FV^{-1}) = \begin{pmatrix} \frac{\alpha_+ \beta_+ \Lambda}{\mu(\beta_+ + \mu)(\gamma_+ + \mu)} & \frac{\alpha_+ \Lambda}{\mu(\gamma_+ + \mu)} \\ 0 & 0 \end{pmatrix} = \frac{\alpha_+ \beta_+ \Lambda}{\mu(\beta_+ + \mu)(\gamma_+ + \mu)}$$

Now, we solve for  $R_{SS}^{02}$ . We let  $\bar{x} = (E_-, S_-)^T$ . From (1), we get  $\mathcal{F}(\bar{x})$  and  $\mathcal{V}(\bar{x})$  where:

$$\mathcal{F}(\bar{x}) = \begin{pmatrix} \alpha_- IS_- \\ 0 \end{pmatrix} \text{ and } \mathcal{V}(\bar{x}) = \begin{pmatrix} \beta_- E_- + \mu E_- \\ -\beta_- E_- + \gamma_- S_- + \delta_- S_- S_+ + \mu S_- \end{pmatrix}. \quad (4)$$

From (4), we obtain  $F = \begin{pmatrix} 0 & \frac{\alpha_- \Lambda}{\mu} \\ 0 & 0 \end{pmatrix}$  and  $V = \begin{pmatrix} \beta_- + & 0 \\ -\beta_- & \gamma_- + \mu \end{pmatrix}$ .

Therefore, the reproduction number of rumor  $R_{SS}^{02}$  is the largest eigenvalue of  $FV^{-1}$  shown below:

$$R_{SS}^{02} = \rho(FV^{-1}) = \begin{pmatrix} \frac{\alpha_- \beta_- \Lambda}{\mu(\beta_- + \mu)(\gamma_- + \mu)} & \frac{\alpha_- \Lambda}{\mu(\gamma_- + \mu)} \\ 0 & 0 \end{pmatrix} = \frac{\alpha_- \beta_- \Lambda}{\mu(\beta_- + \mu)(\gamma_- + \mu)}.$$

**Theorem 1.** The truth-rumor-free equilibrium  $E_{SS}^0$  is locally asymptotically stable whenever  $R_{SS}^0 < 1$ .

**Proof.** Take the Jacobian matrix at  $E^0_{SS}$ :

$$J(E_{SS}^0) = \begin{pmatrix} -\mu & 0 & 0 & -\frac{\alpha_+ \Lambda}{\mu} & -\frac{\alpha_- \Lambda}{\mu} \\ 0 & -\beta_+ - \mu & 0 & \frac{\alpha_+ \Lambda}{\mu} & 0 \\ 0 & 0 & -\beta_- - \mu & 0 & \frac{\alpha_- \Lambda}{\mu} \\ 0 & \beta_+ & 0 & -\gamma_+ - \mu & 0 \\ 0 & 0 & \beta_- & 0 & -\gamma_- - \mu \end{pmatrix}$$

whose characteristic polynomial is:

$$M(\lambda) = (\lambda + \mu)M_1(\lambda)M_2(\lambda)$$

where:

$$M_1(\lambda) = \lambda^2 + a_1\lambda + a_2 \quad (5), \text{ and}$$

$$M_2(\lambda) = \lambda^2 + b_1\lambda + b_2. \quad (6)$$

In (5),

$$a_1 = \beta_+ + \gamma_+ + 2\mu > 0, \text{ and}$$

$$a_2 = (\beta_+ + \mu)(\gamma_+ + \mu) - \frac{\alpha_+ \beta_+ \Lambda}{\mu}.$$

Since  $R_{SS}^0 < 1$ , then  $R_{SS}^{01} < 1$ . We have:

$$\begin{aligned} 1 - R_{SS}^{01} > 0 &\Rightarrow 1 - \frac{\alpha_+ \beta_+ \Lambda}{\mu(\beta_+ + \mu)(\gamma_+ + \mu)} > 0 \\ &\Rightarrow (\beta_+ + \mu)(\gamma_+ + \mu) - \frac{\alpha_+ \beta_+ \Lambda}{\mu} > 0 \\ &\Rightarrow a_2 > 0. \end{aligned}$$

In (6),

$$b_1 = \beta_- + \gamma_- + 2\mu > 0, \text{ and}$$

$$b_2 = (\beta_- + \mu)(\gamma_- + \mu) - \frac{\alpha_- \beta_- \Lambda}{\mu}.$$

Since  $R_{SS}^0 < 1$ , then  $R_{SS}^{02} < 1$ . We have,

$$\begin{aligned} 1 - R_{SS}^{02} > 0 &\Rightarrow 1 - \frac{\alpha_- \beta_- \Lambda}{\mu(\beta_- + \mu)(\gamma_- + \mu)} > 0 \\ &\Rightarrow (\beta_- + \mu)(\gamma_- + \mu) - \frac{\alpha_- \beta_- \Lambda}{\mu} > 0 \\ &\Rightarrow b_2 > 0. \end{aligned}$$

By Routh-Hurwitz criterion, the roots of  $M_1(\lambda)$  and  $M_2(\lambda)$  are of negative real parts. Therefore,  $E_{SS}^0$  is locally asymptotically stable whenever  $R_{SS}^0 < 1$ . ■

**Theorem 2.** The truth-rumor-free equilibrium  $E_{SS}^0$  is globally asymptotically stable whenever  $\Lambda(\alpha_+ + \alpha_-) \leq 2\mu^2$ .

**Proof.** Take the Lyapunov function  $L = E_+(t) + E_-(t) + S_+(t) + S_-(t) + R(t)$ . Then,  $L' = (\alpha_+ I - \mu)S_+ + (\alpha_- I - \mu)S_- - \mu(E_+ + E_- + R)$ .

Since  $I \leq \frac{\Lambda}{\mu}$  and  $\Lambda(\alpha_+ + \alpha_-) \leq 2\mu^2$ ,

$$L' \leq \left(\alpha_+ \frac{\Lambda}{\mu} - \mu\right)S_+ + \left(\alpha_- \frac{\Lambda}{\mu} - \mu\right)S_- - \mu(E_+ + E_- + R) \leq 0.$$

Since  $\mu > 0$ , then  $L'(t) \leq 0$ . And, equality only holds if  $E_+(t) = E_-(t) = S_+(t) = S_-(t) = R(t) = 0$ .  $E_{SS}^0$  is the only solution of (1) where this occurs. Thus, all solutions of (1) approach  $E_{SS}^0$  as  $t \rightarrow \infty$ . Therefore,  $E_{SS}^0$  is globally asymptotically stable. ■

*The existence of the truth-free equilibrium and rumor-free equilibrium.* Assuming that  $E_{SS}^T(I_{SS}^T, E_{+SS}^T, E_{-SS}^T, S_{+SS}^T, S_{-SS}^T, R_{SS}^T)$  is the steady-state of (1) and  $E_{+SS}^T, S_{+SS}^T = 0$ , then it satisfies the following equations:

$$\begin{cases} 0 &= \Lambda - \alpha_- I S_- - \mu I \\ 0 &= \alpha_- I S_- - \beta_- E_- - \mu E_- \\ 0 &= \beta_- E_- - \gamma_- S_- - \mu S_- \\ 0 &= \gamma_- S_- - \mu R \end{cases} \quad (7)$$

Solving the first and second equations of (7) gives us:

$$I_{SS}^T = \frac{\Lambda}{\alpha_- S_{-SS}^T + \mu} \text{ and } E_{-SS}^T = \frac{\alpha_- I_{SS}^T S_{-SS}^T}{\beta_- + \mu} \quad (8)$$

Substituting the above equations into the third equation of (7) gives us:

$$S_{-SS}^T = \frac{\beta_- \Lambda}{(\beta_- + \mu)(\gamma_- + \mu)} - \frac{\mu}{\alpha_-}$$

In terms of  $R_{SS}^{02}$ , we can represent  $S_{-SS}^T$  as  $S_{-SS}^T = \frac{\mu}{\alpha_-} (R_{SS}^{02} - 1)$ . We can summarize the above into the following theorem.

**Theorem 3.** There exists a truth-free equilibrium  $E_{SS}^T(I_{SS}^T, 0, E_{-SS}^T, 0, S_{-SS}^T, R_{SS}^T)$ , where:

$$I_{SS}^T = \frac{\Lambda}{\mu R_{SS}^{02}}, E_{-SS}^T = \frac{\Lambda(1 - (R_{SS}^{02})^{-1})}{\beta_- + \mu}, S_{-SS}^T = \frac{\mu}{\alpha_-} (R_{SS}^{02} - 1), R_{SS}^T = \frac{\gamma_-}{\alpha_-} (R_{SS}^{02} - 1)$$

We can obtain a similar theorem for the rumor-free equilibrium together with the following assumption.  $E_{SS}^R(I_{SS}^R, E_{+SS}^R, E_{-SS}^R, S_{+SS}^R, S_{-SS}^R, R_{SS}^R)$  is the steady-state of (1) and  $E_{SS}^R, S_{-SS}^R = 0$ , then it satisfies the following equations:

$$\begin{cases} 0 &= \Lambda - \alpha_+ I S_+ - \mu I \\ 0 &= \alpha_+ I S_+ - \beta_+ E_+ - \mu E_+ \\ 0 &= \beta_+ E_+ - \gamma_- S_+ - \mu S_+ \\ 0 &= \gamma_+ S_+ - \mu R \end{cases}$$

**Theorem 4.** There exists a rumor-free equilibrium  $E_{SS}^R(I_{SS}^R, E_{+SS}^R, 0, S_{+SS}^R, 0, R_{SS}^R)$  where:

$$I_{SS}^R = \frac{\Lambda}{\mu R_{SS}^{01}}, E_{+SS}^R = \frac{\Lambda(1 - (R_{SS}^{01})^{-1})}{\beta_+ + \mu}, S_{+SS}^R = \frac{\mu}{\alpha_+} (R_{SS}^{01} - 1), R_{SS}^R = \frac{\gamma_+}{\alpha_+} (R_{SS}^{01} - 1)$$

**Theorem 5.** The truth-free equilibrium  $E_{SS}^T$  is locally asymptotically stable whenever  $R_{SS}^{01} < 1$  and  $R_{SS}^{02} > 1$ .

**Proof.** Take the Jacobian matrix at  $E_{SS}^T$ :

$$J(E_{SS}^T) = \begin{pmatrix} -\mu(R_{SS}^{02} - 1) - \mu & 0 & 0 & -\frac{\alpha_+ \Lambda}{\mu} R_{SS}^{02} & -\frac{\alpha_- \Lambda}{\mu} R_{SS}^{02} \\ 0 & -\beta_+ - \mu & 0 & \frac{\alpha_+ \Lambda}{\mu} R_{SS}^{02} & 0 \\ \mu(R_{SS}^{02} - 1) & 0 & -\beta_- - \mu & 0 & \frac{\alpha_- \Lambda}{\mu} R_{SS}^{02} \\ 0 & \beta_+ & 0 & -\gamma_+ - \frac{\delta_+ \mu}{\alpha_-} (R_{SS}^{02} - 1) - \mu & 0 \\ 0 & 0 & \beta_- & -\frac{\delta_- \mu}{\alpha_-} (R_{SS}^{02} - 1) & -\gamma_- - \mu \end{pmatrix}$$

whose characteristic polynomial is:

$$N(\lambda) = N_1(\lambda)N_2(\lambda)$$

where:

$$N_1(\lambda) = \lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 \quad (9)$$

$$N_2(\lambda) = \lambda^2 + d_1\lambda + d_2. \quad (10)$$

Note that  $R_{SS}^{02} > 1$ . In (9):

$$c_1 = \mu R_{SS}^{02} + \beta_- + \gamma_- + 2\mu > 0$$

$$c_3 = \alpha_- \beta_- \Lambda (1 - (R_{SS}^{02})^{-1}) > 0$$

Observe that:

$$\left( \frac{1}{\gamma_- + \mu} + \frac{1}{\beta_- + \mu} \right) \left( \beta_- + \gamma_- + 2\mu + \frac{\alpha_- \beta_- \Lambda}{(\beta_- + \mu)(\gamma_- + \mu)} \right)$$

$$= \frac{(\beta_- + \gamma_- + \mu)^2}{(\beta_- + \mu)(\gamma_- + \mu)} + \frac{(\beta_- + \gamma_- + \mu)(\alpha_- \beta_- \Lambda)}{(\beta_- + \mu)^2(\gamma_- + \mu)^2} > \frac{(\beta_- + \gamma_- + \mu)^2}{(\beta_- + \mu)(\gamma_- + \mu)} > 1 > 1 - (R_{SS}^{02})^{-1}$$

In particular:

$$\left( \frac{1}{\gamma_- + \mu} + \frac{1}{\beta_- + \mu} \right) \left( \beta_- + \gamma_- + 2\mu + \frac{\alpha_- \beta_- \Lambda}{(\beta_- + \mu)(\gamma_- + \mu)} \right) > 1 - (R_{SS}^{02})^{-1}$$

which implies that:

$$\begin{aligned} & \left( \frac{\alpha_- \beta_- \Lambda}{\gamma_- + \mu} + \frac{\alpha_- \beta_- \Lambda}{\beta_- + \mu} \right) \left( \beta_- + \gamma_- + 2\mu + \frac{\alpha_- \beta_- \Lambda}{(\beta_- + \mu)(\gamma_- + \mu)} \right) \\ & > \alpha_- - \beta_- - \Lambda - \mu(\beta_- + \mu)(\gamma_- + \mu) \\ & \Rightarrow c_1 c_2 > c_3 \Rightarrow c_1 c_2 - c_3 > 0 \end{aligned}$$

In (10):

$$\begin{aligned} d_1 &= \beta_+ + \gamma_+ + \frac{\delta_+ \mu}{\alpha_-} (R_{SS}^{02} - 1) + 2\mu > 0 \\ d_2 &= (\beta_+ + \mu) \left( \gamma_+ + \frac{\delta_+ \mu}{\alpha_-} (R_{SS}^{02} - 1) + \mu \right) - \frac{\alpha_+ \beta_+ \Lambda}{\mu} (R_{SS}^{02})^{-1} \end{aligned}$$

Since  $R_{SS}^{02} > 1$ , then:=

$$\begin{aligned} (\beta_+ + \mu) \left( \gamma_+ + \frac{\delta_+ \mu}{\alpha_-} (R_{SS}^{02} - 1) + \mu \right) &> (\beta_+ + \mu) \left( \gamma_+ + \frac{\delta_+ \mu}{\alpha_-} (1 - 1) + \mu \right) \quad (11), \text{ and} \\ &= (\beta_+ + \mu)(\gamma_+ + \mu) \end{aligned}$$

$$\begin{aligned} 0 < (R_{SS}^{02})^{-1} < 1 &\Rightarrow \frac{\alpha_+ \beta_+ \Lambda}{\mu} (R_{SS}^{02})^{-1} < \frac{\alpha_+ \beta_+ \Lambda}{\mu} \quad (12) \\ &\Rightarrow -\frac{\alpha_+ \beta_+ \Lambda}{\mu} (R_{SS}^{02})^{-1} > -\frac{\alpha_+ \beta_+ \Lambda}{\mu} \end{aligned}$$

Since  $R_{SS}^{01} < 1$ , we have:

$$\begin{aligned} 1 - R_{SS}^{01} > 0 &\Rightarrow 1 - \frac{\alpha_+ \beta_+ \Lambda}{\mu(\beta_+ + \mu)(\gamma_+ + \mu)} > 0 \quad (13) \\ &\Rightarrow (\beta_+ + \mu)(\gamma_+ + \mu) - \frac{\alpha_+ \beta_+ \Lambda}{\mu} > 0 \end{aligned}$$

Combining (11), (12), and (13), we obtain:

$$d_2 > (\beta_+ + \mu)(\gamma_+ + \mu) - \frac{\alpha_+ \beta_+ \Lambda}{\mu} > 0$$

By Routh-Hurwitz criterion, the roots of  $N_1(\lambda)$  and  $N_2(\lambda)$  are of negative real parts. Therefore,  $E_{SS}^T$  is locally asymptotically stable whenever  $R_{SS}^{01} < 1$  and  $R_{SS}^{02} > 1$ . ■

**Theorem 6.** The rumor-free equilibrium  $E_{SS}^R$  is locally asymptotically stable whenever  $R_{SS}^{01} > 1$  and  $R_{SS}^{02} < 1$ .

**Proof.** The proof is similar to the proof of Theorem 5.

**The existence of the truth-rumor coexistence equilibrium.** Assuming that  $E_{SS}^*(I_{SS}^*, E_{+SS}^*, E_{-SS}^*, S_{+SS}^*, S_{-SS}^*, R_{SS}^*)$  is the steady states of the system (1), then it satisfies the following equations:

$$\begin{cases} 0 = \Lambda - \alpha_+ I S_+ - \alpha_- I S_- - \mu I \\ 0 = \alpha_+ I S_+ - \beta_+ E_+ - \mu E_+ \\ 0 = \alpha_- I S_- - \beta_- E_- - \mu E_- \\ 0 = \beta_+ E_+ - \gamma_+ S_+ - \delta_+ S_+ S_- - \mu S_+ \\ 0 = \beta_- E_- - \gamma_- S_- - \delta_- S_- S_+ - \mu S_- \\ 0 = \gamma_+ S_+ + \delta_+ S_+ S_- + \gamma_- S_- + \delta_- S_- S_+ - \mu R \end{cases} \quad (14)$$

From the second and third equations of (14), we obtain:

$$E_{+SS}^* = \frac{\alpha_+ I_{SS}^* S_{+SS}^*}{\beta_+ + \mu} \text{ and } E_{-SS}^* = \frac{\alpha_- I_{SS}^* S_{-SS}^*}{\beta_- + \mu}. \quad (15)$$

Substituting (15) into the fourth and fifth equations of (14), we have:

$$S_{+SS}^* = \frac{\alpha_- \beta_- I_{SS}^* - (\beta_- + \mu)(\gamma_- + \mu)}{\delta_- (\beta_- + \mu)} \text{ and } S_{-SS}^* = \frac{\alpha_+ \beta_+ I_{SS}^* - (\beta_+ + \mu)(\gamma_+ + \mu)}{\delta_+ (\beta_+ + \mu)} \quad (16)$$

Substituting (15) into the first equation of (14), we find that:

$$F(I) = A(I_{SS}^*)^2 + B(I_{SS}^*) + C$$

where  $A = \alpha_+ \alpha_- \left( \frac{\beta_+ \gamma_-}{\beta_+ + \mu} + \frac{\beta_- \gamma_+}{\beta_- + \mu} \right)$ ,  $B = \mu \gamma_+ \gamma_- - \gamma_- (\gamma_+ + \mu) - \gamma_+ (\gamma_- + \mu)$  and  $C = -\Lambda \gamma_+ \gamma_-$ . Since  $A > 0, C < 0$ , then there exists a unique positive solution  $I_{SS}^*$  such that  $F(I_{SS}^*) = 0$ .

**Theorem 7.** The truth-rumor coexistence equilibrium  $E_{SS}^*$  is unstable whenever it exists.

**Proof.** Take the Jacobian matrix at  $E_{SS}^*$ :

$$\begin{pmatrix} -\alpha_+ S_{+SS}^* - \alpha_- S_{-SS}^* - \mu & 0 & 0 & -\alpha_+ I_{SS}^* & -\alpha_- I_{SS}^* \\ \alpha_+ S_{+SS}^* & -\beta_+ - \mu & 0 & \alpha_+ I_{SS}^* & 0 \\ \alpha_- S_{-SS}^* & 0 & -\beta_- - \mu & 0 & \alpha_- I_{SS}^* \\ 0 & \beta_+ & 0 & -\gamma_+ - \delta_+ S_{-SS}^* - \mu & -\delta_+ S_{+SS}^* \\ 0 & 0 & \beta_- & -\delta_- S_{-SS}^* & -\gamma_- - \delta_- S_{+SS}^* - \mu \end{pmatrix}$$

whose characteristic polynomial is:

$$P(\lambda) = \lambda^5 + e_1 \lambda^4 + e_2 \lambda^3 + e_3 \lambda^2 + e_4 \lambda + e_5$$

where:

$$e_5 = -I_{SS}^* S_{+SS}^* S_{-SS}^* \alpha_+ \alpha_- \beta_+ \delta_- (\beta_- + \mu) < 0.$$

From the Routh-Hurwitz criterion, there exists an eigenvalue with positive real part. Thus,  $E_{SS}^*$  is unstable whenever it exists. ■

### Analysis of Exposed-Spreader Interaction Model

**Truth-and-Rumor-free equilibrium and the basic reproduction number.** We can obtain the Truth-and-Rumor-free equilibrium,  $E_{ES}^0$ , by setting the left side of (2) to 0. It is easy to see that we obtain  $E_{ES}^0 = (\frac{\Lambda}{\mu}, 0, 0, 0, 0)$ . Now, we use the next generation matrix to find the basic reproduction number  $R_{ES}^0$ . In this case,  $R_{ES}^0$  is defined to be the anticipated new generation of truth-rumor spreaders influenced by a member of the spreader population. Since we have two different kinds of spreaders and, hence, information spread, then we solve the reproduction number of truth  $R_{ES}^{01}$  and the reproduction number of rumor  $R_{ES}^{02}$  separately. Then, we set  $R_{ES}^0 = \max\{R_{ES}^{01}, R_{ES}^{02}\}$ .

Now, we solve for  $R_{ES}^{01}$ . We let  $\bar{x} = (E_+, S_+)^T$ . From (2), we obtain  $\mathcal{F}(\bar{x})$  and  $\mathcal{V}(\bar{x})$  where

$$\mathcal{F}(\bar{x}) = \begin{pmatrix} \alpha_+ I S_+ + \epsilon_- E_- S_+ \\ 0 \end{pmatrix} \text{ and } \mathcal{V}(\bar{x}) = \begin{pmatrix} \beta_+ E_+ + \epsilon_+ E_+ S_- + \mu E_+ \\ -\beta_+ E_+ + \gamma_+ S_+ + \mu S_+ \end{pmatrix} \quad (17)$$

From (17), we obtain:

$$F = \begin{pmatrix} 0 & \frac{\alpha_+ \Lambda}{\mu} \\ 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} \beta_+ + & 0 \\ -\beta_+ & \gamma_+ + \mu \end{pmatrix}$$

From here, we can see that the reproduction number of truth  $R_{ES}^{01} = R_{SS}^{01}$ .

Now, we solve for  $R_{ES}^{02}$ . We let  $X = (E_-, S_-)^T$ . From (2), we obtain  $\mathcal{F}(\bar{x})$  and  $\mathcal{V}(\bar{x})$  where:

$$\mathcal{F}(\bar{x}) = \begin{pmatrix} \alpha_- I S_- + \epsilon_+ E_+ S_- \\ 0 \end{pmatrix} \text{ and } \mathcal{V}(\bar{x}) = \begin{pmatrix} \beta_- E_- + \epsilon_- E_- S_+ + \mu E_- \\ -\beta_- E_- + \gamma_- S_- + \mu S_- \end{pmatrix} \quad (18)$$

From (18), we obtain:

$$F = \begin{pmatrix} 0 & \frac{\alpha_- \Lambda}{\mu} \\ 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} \beta_- + & 0 \\ -\beta_- & \gamma_- + \mu \end{pmatrix}$$

From here, we can see that the reproduction number of truth  $R_{ES}^{02} = R_{SS}^{02}$ .

**Theorem 8.** The truth-rumor-free equilibrium  $E_{ES}^0$  is locally asymptotically stable whenever  $R_{ES}^0 < 1$ .

**Proof.** Take the Jacobian matrix at  $E_{ES}^0$ :

$$J(E_{SS}^0) = \begin{pmatrix} -\mu & 0 & 0 & -\frac{\alpha_+ \Lambda}{\mu} & -\frac{\alpha_- \Lambda}{\mu} \\ 0 & -\beta_+ - \mu & 0 & \frac{\alpha_+ \Lambda}{\mu} & 0 \\ 0 & 0 & -\beta_- - \mu & 0 & \frac{\alpha_- \Lambda}{\mu} \\ 0 & \beta_+ & 0 & -\gamma_+ - \mu & 0 \\ 0 & 0 & \beta_- & 0 & -\gamma_- - \mu \end{pmatrix}.$$

From here, it is easy to see that the rest of the proof is similar to the proof of Theorem 1.

**Theorem 9.** The truth-rumor-free equilibrium  $E_{ES}^0$  is globally asymptotically stable whenever  $\Lambda(\alpha_+ + \alpha_-) \leq 2\mu^2$ .

**Proof.** Take the Lyapunov function:  $L = E_+(t) + E_-(t) + S_+(t) + S_-(t) + R(t)$ . Then:

$$L' = (\alpha_+ - \mu)S_+ + (\alpha_- - \mu)S_- - \mu(E_+ + E_- + R)$$

From here, it is easy to see that the rest of the proof is similar to the proof of Theorem 2.

**The existence of the truth-free equilibrium and rumor-free equilibrium.** Assuming that  $E_{ES}^T(I_{ES}^T, E_{+ES}^T, E_{-ES}^T, S_{+ES}^T, S_{-ES}^T, R_{ES}^T)$  is the steady-state of (2) and let  $E_{+SS}^T, S_{+SS}^T = 0$ , then it satisfies the following equations:

$$\begin{cases} 0 &= \Lambda - \alpha_- I S_- - \mu I \\ 0 &= \alpha_- I S_- - \beta_- E_- - \mu E_- \\ 0 &= \beta_- E_- - \gamma_- S_- - \mu S_- \\ 0 &= \gamma_- S_- - \mu R \end{cases} \quad (19)$$

Similarly to Theorem 5, we can conclude that:

**Theorem 10.** There exists a truth-free equilibrium  $E_{ES}^T(I_{ES}^T, 0, E_{-ES}^T, 0, S_{-ES}^T, R_{ES}^T)$  where:

$$I_{ES}^T = \frac{\Lambda}{\mu R_{ES}^{02}}, E_{-ES}^T = \frac{\Lambda(1 - (R_{ES}^{02})^{-1})}{\beta_- + \mu}, S_{-ES}^T = \frac{\mu}{\alpha_-} (R_{ES}^{02} - 1), R_{ES}^T = \frac{\gamma_-}{\alpha_-} (R_{ES}^{02} - 1)$$

We can obtain a similar theorem for the rumor-free equilibrium together with the following assumption.  $E_{ES}^R(I_{ES}^R, E_{+ES}^R, E_{-ES}^R, S_{+ES}^R, S_{-ES}^R, R_{ES}^R)$  is the steady-state of (2) and  $E_{-ES}^R = S_{-ES}^R = 0$ , then it satisfies the following equations:

$$\begin{cases} 0 &= \Lambda - \alpha_+ I S_+ - \mu I \\ 0 &= \alpha_+ I S_+ - \beta_+ E_+ - \mu E_+ \\ 0 &= \beta_+ E_+ - \gamma_+ S_+ - \mu S_+ \\ 0 &= \gamma_+ S_+ + \mu R \end{cases}$$

**Theorem 11.** There exists a rumor-free equilibrium  $E_{ES}^R(I_{ES}^R, E_{+ES}^R, 0, S_{+ES}^R, 0, R_{ES}^R)$  where:

$$I_{ES}^R = \frac{\Lambda}{\mu R_{ES}^{01}}, E_{+ES}^R = \frac{\Lambda(1 - (R_{ES}^{01})^{-1})}{\beta_+ + \mu}, S_{+ES}^R = \frac{\mu}{\alpha_+} (R_{ES}^{01} - 1), R_{ES}^R = \frac{\gamma_+}{\alpha_+} (R_{ES}^{01} - 1)$$

**Theorem 12.** The truth-free equilibrium  $E_{ES}^T$  is locally asymptotically stable whenever  $R_{ES}^{01} < \frac{\alpha_+(\beta_- + \mu)}{\alpha_+(\beta_- + \mu) + \mu\epsilon_-}$  and  $R_{ES}^{02} > 1$ .

**Theorem 13.** The rumor-free equilibrium  $E_{ES}^R$  is locally asymptotically stable whenever  $R_{ES}^{01} > 1$  and  $R_{ES}^{02} < \frac{\alpha_-(\beta_+ + \mu)}{\alpha_-(\beta_+ + \mu) + \mu\epsilon_+}$ .

**The existence of the truth-rumor coexistence equilibrium.** Assuming that  $E_{ES}^*(I_{ES}^*, E_{+ES}^*, E_{-ES}^*, S_{+ES}^*, S_{-ES}^*, R_{ES}^*)$  is the steady states of the system (2), then it satisfies the following equations:

$$\begin{cases} 0 &= \Lambda - \alpha_+ I S_+ - \alpha_- I S_- - \mu I \\ 0 &= \alpha_+ I S_+ + \epsilon_- E_- S_+ - \beta_+ E_+ - \epsilon_+ E_+ S_- - \mu E_+ \\ 0 &= \alpha_- I S_- + \epsilon_+ E_+ S_- - \beta_- E_- - \epsilon_- E_- S_+ - \mu E_- \\ 0 &= \beta_+ E_+ - \gamma_+ S_+ - \mu S_+ \\ 0 &= \beta_- E_- - \gamma_- S_- - \mu S_- \\ 0 &= \gamma_+ S_+ + \gamma_- S_- - \mu R \end{cases} \quad (20)$$

From the second and third equations of (20), we obtain:

$$E_{+ES}^* = \frac{(\alpha_+ \beta_- + \alpha_+ \mu + \alpha_+ \epsilon_- S_{+ES}^* + \alpha_- \epsilon_- S_{-ES}^*) I_{ES}^* S_{+ES}^*}{(\beta_+ + \mu)(\beta_- + \mu) + \epsilon_+ (\beta_- + \mu) S_{-ES}^* + \epsilon_- (\beta_+ + \mu) S_{+ES}^*} \quad (21)$$

and:

$$E_{-ES}^* = \frac{(\alpha_- \beta_+ + \alpha_- \mu + \alpha_- \epsilon_+ S_{-ES}^* + \alpha_+ \epsilon_+ S_{+ES}^*) I_{ES}^* S_{-ES}^*}{(\beta_+ + \mu)(\beta_- + \mu) + \epsilon_+ (\beta_- + \mu) S_{-ES}^* + \epsilon_- (\beta_+ + \mu) S_{+ES}^*} \quad (22)$$

Substituting (21) and (22) into the fourth and fifth equations of (20), we have:

$$S_{+SS}^* = \frac{\beta_+ (\alpha_- \beta_- I_{ES}^* - (\beta_- + \mu)(\gamma_- + \mu))}{\beta_+ \epsilon_- (\gamma_- + \mu) - \beta_- \epsilon_+ (\gamma_+ + \mu)} \quad (23)$$

and

$$S_{-SS}^* = \frac{\beta_- (\alpha_+ \beta_+ I_{ES}^* - (\beta_+ + \mu)(\gamma_+ + \mu))}{\beta_- \epsilon_+ (\gamma_+ + \mu) - \beta_+ \epsilon_- (\gamma_- + \mu)} \quad (24)$$

Substituting (23) and (24) into the first equation of (20), we find that:

$$G(I) = Q(I_{ES}^*) + R$$

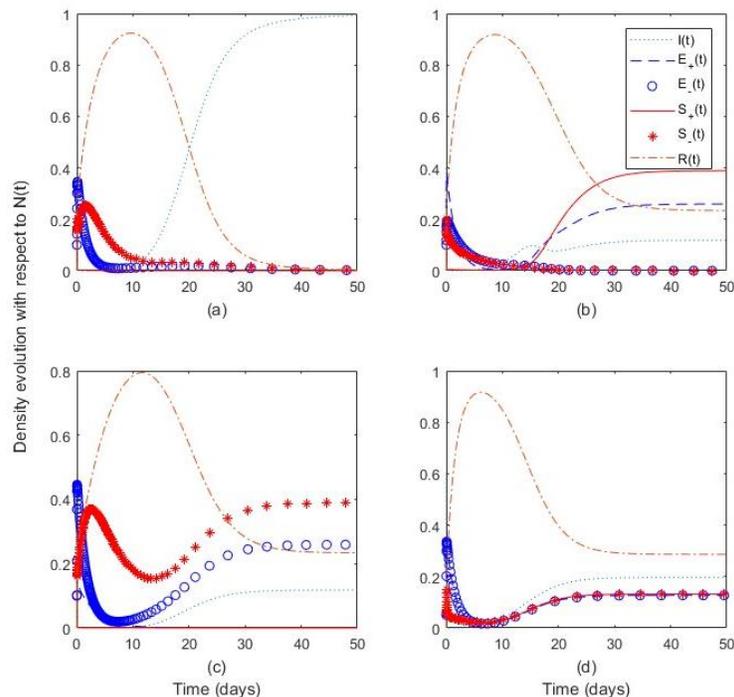
where  $Q = \alpha_+ \beta_+ (\beta_- + \mu)(\gamma_- + \mu) - \alpha_- \beta_- (\beta_+ + \mu)(\gamma_+ + \mu) - \mu(\beta_+ \epsilon_- (\gamma_- + \mu) - \beta_- \epsilon_+ (\gamma_+ + \mu))$ ,  $R = \Lambda(\beta_+ \epsilon_- (\gamma_- + \mu) - \beta_- \epsilon_+ (\gamma_+ + \mu))$ . Then, there exists a unique positive solution  $I_{SS}^*$  such that  $G(I_{SS}^*) = 0$ .

## NUMERICAL SIMULATIONS

In this section, we show simulations that illustrate the ODE models and results of the discussions found above. We decided upon the assigned values based on the calculated reproduction numbers in the previous section.

### Spreader-Spreader Model: Removing Rumor Spreaders to Only Allow Truth to Persist

Using the spreader-spreader model, we simulated a scenario with the following initial conditions: 400 Ignorant, 100 Exposed to Truth, 100 Exposed to Rumors, 200 Spreader of Truth, and 200 Spreader of Rumors. It is designed to show how truth or rumor can die out despite having many exposed to Spreaders of Truth and Rumors.



**Figure 3.** Numerical simulations of the spreader-spreader model that depicts the different equilibrium points. (a) Truth-Rumor-Free equilibrium was achieved by using the parameter values that produce reproduction numbers  $R_{SS}^{01} = 0.778 < 1$  and  $R_{SS}^{02} = 0.778 < 1$  for truth and rumor, respectively. (b) Rumor-Free equilibrium is achieved by having reproduction numbers  $R_{SS}^{01} = 8.471 > 1$  and  $R_{SS}^{02} = 0.671 < 1$ . (c) Truth-Free equilibrium is achieved with reproduction numbers  $R_{SS}^{01} = 0.671 < 1$  and  $R_{SS}^{02} = 8.471 > 1$ . (d) Lastly, an equilibrium where truth and rumor coexists is achieved when both reproduction numbers are greater than one —  $R_{SS}^{01} = 7.096 > 1$  and  $R_{SS}^{02} = 7.096 > 1$ .

Suppose we have a scenario where we want to keep all things confidential and top secret and we do not want any truth or rumor to spread. For this to happen, the reproduction of truth and rumor has to be less than one as required by Theorem 1. Following the parameters for Truth-Rumor-Free equilibrium in Appendix II Table 1, the resulting reproduction numbers for both truth and rumor are 0.778, which are less than 1. Theorem 2 and Figure 3(a) show that both truth and rumors would die out eventually.

Second, suppose we have a scenario where we have affected policymakers or agencies who want to spread correct details about a topical event that has been subject to rumoring. Here, we want the truth to persist and the rumors to disappear. Theorem 6 would require us to have the reproduction number for truth to be greater than 1 and the reproduction number for rumors to be less than 1. Using the spreading rates for Rumor-Free in Appendix II Table 1, we produce a reproduction number for truth that is 8.471, which is greater than 1, and a reproduction number for rumors that is 0.671, which is less than 1. Figure 3(b) shows the plot of the simulation where the truth persists and the rumors die out eventually.

Third, we simulate a scenario where a multitude of fake news drowns out verified facts about a topical event. Here, truthful information disappears and the rumors to persist. According to Theorem 1, we would want the reproduction number for the truth to be less than 1 and the reproduction number for rumors to be greater than 1. Using the parameters for Truth-Free equilibrium in Appendix II Table 1 would give us 0.671 as the reproduction number for truth and is less than 1, and 8.471 for rumor that is greater than 1. Figure 3(c) shows the plot of the simulation where the truth dies out eventually and the rumors persist.

Fourth, we simulate a scenario where we want truth and rumor to coexist. Real-life examples of this scenario include are publicity agents who want exposure and awareness for their clients so they spread rumors to pique people's curiosity. In this situation, we want the reproduction numbers for both truth and rumor to be greater than 1. For this simulation, we used a different distribution of actors (600 who are Ignorant, 50 each for Exposed to Truth and Rumor, and 150 for Spreaders of Truth and Rumor) because we want to see if both truth and rumors still persist despite having fewer spreaders. To obtain the reproduction number necessary for this scenario, we used the rates for spreading for Coexistence in Appendix II Table 1. This gives us 7.096 as the reproduction number for both truth and rumor. Figure 3(d) shows how both truth and rumors exist.

**Varying the effectiveness of truth spreaders.** Controlling the spread of rumor is a non-trivial endeavor especially in critical situations. It is close to impossible to suppress rumors because of the overwhelming number of people driven by anxiety and ambiguity of information available to them. It is more plausible to focus on spreading truth through legitimate means of information dissemination. To test this, we ran simulations to determine the effects of changing the rate of effectiveness of spreaders to spread the truth. Particularly, we investigated the effects of changing the rate of effectiveness of spreaders to spread the truth to the ignorant people ( $\alpha_+$ ) and effectiveness of spreaders to convince Spreaders of Rumors to stop spreading them ( $\delta_-$ ). To simulate this scenario, we set 10 spreaders each for truth and rumor, and 980 who are ignorant because we want to see if the truth or rumor will spread despite having a few spreaders.

To see the result of changing the effectiveness of spreaders to educate ignorant people with the truth ( $\alpha_+$ ), we let the value of  $\alpha_+$  take on values from [0.2, 0.8] with increments of 0.2 while the other parameters take on values for the Varying  $\alpha_+$  scenario in Appendix II Table 2.

From Appendix III Figure 1, increasing the effectiveness of spreaders to educate the Ignorant people with the truth ( $\alpha_+$ ) promotes the spread of truth and controls the spread of rumors. It can be seen in Appendix III Figure 1(a) that the proportion of Ignorant people in the population decreases as the effectiveness of Spreaders of Truth to educate the Ignorant people increases.

In Appendix III Figure 1(b), the proportion of removed increases when  $\alpha_+$  is at 0.4. When  $\alpha_+$  is at 0.6 and 0.8, the proportion of removed stays relatively the same compared to when  $\alpha_+$  is at 0.2.

In Figures 4(c) to 4(f), we find that the truth fails to spread and dies out while rumors continuously spread until it persists. Here, the Spreaders of Rumor are more effective at spreading rumors to Ignorant people than the Spreaders of Truth at spreading truth ( $\alpha_+ = 0.2 < 0.3 = \alpha_-$ ). The truth continuously spreads until it persists while rumors fail to spread and dies out when the Spreaders of Truth are better than the Spreaders of Rumors in terms of the effectiveness of spreading ( $\alpha_+ = 0.4, 0.6, 0.8 > 0.3 = \alpha_-$ ). However, there is little difference when we increase  $\alpha_+$  from 0.4 to 0.6 and even less when we increase it from 0.6 to 0.8. If we only want the truth to remain and the rumors to die out, it is enough to increase  $\alpha_+$  just above  $\alpha_-$ . Increasing  $\alpha_+$  further only gives us diminishing returns. Although, we would still want to increase the effectiveness of Spreaders of Truth to spread the truth if we want to spread the truth to more people.

Similarly, to see the effects of changing the effectiveness of Spreaders of Truth to convince Spreaders of Rumors to stop spreading ( $\delta_-$ ), we perform simulations where  $\delta_-$  takes on values from [0.2, 0.8] with increments of 0.2 and use the values for the Varying  $\delta_-$  in Appendix II Table 2.

Appendix III Figure 2 shows that increasing the effectiveness of Spreaders of Truth to convince the Spreaders of Rumors to stop spreading rumors ( $\delta_-$ ) benefits the spread of truth and impedes the spread of rumors. It can be seen in Appendix III Figure 2(a) that the proportion of Ignorant people remains the same despite any increase or decrease in the effectiveness of the Spreaders of Truth.

As observed in Appendix III Figure 2(b), the proportion of removed decreases during the first few days as the effectiveness of Spreaders of Truth to educate those exposed to rumor increases. Yet, the proportion of removed

converges as more days pass. Spreaders of Truth would need to be more effective during those first few days to spread more truth.

Examining Appendix III Figures 2(c) to 2(f), the truth fails to spread and dies out while rumors continue to spread until rumors persist. This happens when the Spreaders of Rumor are more effective at convincing the Spreaders of Truth to stop spreading than the other way ( $\delta_- = 0.2 < 0.3 = \delta_+$ ). The truth spreads until it persists while rumors fail to spread and die out when the Spreaders of Truth are more effective at convincing the Spreaders of Rumors to stop spreading than the Spreaders of Rumors at convincing the Spreaders of Truth to stop spreading ( $\delta_- = 0.4, 0.6, 0.8 > 0.3 = \delta_+$ ) which is similar to the results of varying  $\alpha_+$ .

The increase in the proportion of Exposed to and Spreaders of Truth decreases when we increase the effectiveness of Spreaders of Truth at convincing the Spreaders of Rumors to stop spreading from 0.4 to 0.6 and from 0.6 to 0.8. This emphasizes the findings in Appendix III Figures 1(c) to 1(f) that it is sufficient to increase the effectiveness of the Spreaders of Truth at convincing the Spreaders of Rumors to stop spreading a little over the effectiveness of the Spreaders of Rumors at convincing the Spreaders of Truth to stop spreading.

The difference between Appendix III Figure 2 and Appendix III Figure 1 can be seen in Appendix III Figure 2(f). During the first 10 days, there is a considerably lower proportion of Spreaders of Rumors when the effectiveness of Spreaders of Truth at convincing the Spreaders of Rumors to stop spreading is increased. This maintains the need for Spreaders of Truth to be more effective early in the diffusion process.

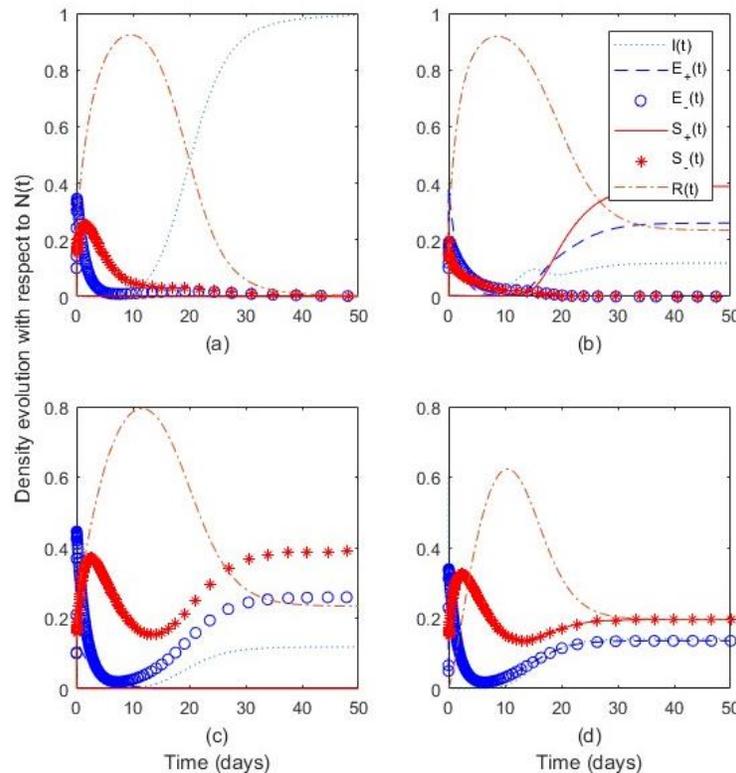
#### **Exposed-Spreader Model: Propagating Truth by Spreading It among Ignorant People and Exposed to Rumors**

In the exposed-spreader model, we simulated scenarios with 400 Ignorant, 100 Exposed to Truth, 100 Exposed to Rumors, 200 Truth Spreaders, and 200 Rumor Spreaders. It is designed to show that the truth or rumors will die out despite having many spreaders for truth and rumors.

If we want truth and rumor to die out [Figure 4(a)], the reproduction numbers for truth and rumor have to be less than 1 by Theorem 8. We use the parameters for Truth-Rumor-Free in Appendix II Table 3 to satisfy this condition.

If we only want the truth to persist and the rumors to die out [Figure 4(b)], Theorem 13 requires the reproduction number for the truth to be greater than 1 and the reproduction number for rumor to be less than 1. The values for Rumor-Free on Appendix II Table 3 give us the desired reproduction numbers.

Theorem 12 gives us the necessary conditions for the truth to die out and rumors to persist [Figure 4(c)]. For this, the reproduction number for truth has to be less than 1 and the reproduction number for rumor to be greater than 1. We use the values for Truth-Free equilibrium found in Appendix II Table 3.



**Figure 4.** Numerical simulations of the exposed-spreader model showing different equilibrium points. (a) The Truth-Rumor-Free equilibrium was achieved with reproduction numbers  $R_{ES}^{01} = 0.778 < 1$  and  $R_{ES}^{02} = 0.778 < 1$  for truth and rumor, respectively. (b) Rumor-Free equilibrium is achieved by having reproduction numbers  $R_{ES}^{01} = 10.487 > 1$  and  $R_{ES}^{02} = 0.646 < 1$ . (c) Truth-Free equilibrium exists if  $R_{ES}^{01} = 10.487 > 1$  and  $R_{ES}^{02} = 0.646 < 1$ . (d) Lastly, truth and rumor coexist then  $R_{ES}^0 > 1$  such as  $R_{ES}^{01} = 15.294$  and  $R_{ES}^{02} = 15.294$ .

We can make both truth and rumor persist (Figure 4(d)) if the reproduction numbers for both truth and rumors are greater than 1. We simulate this with an initial population of 600 Ignorant people, 50 Exposed to Truth, 50 Exposed to Rumor, 150 Spreaders of Truth and 150 Spreaders of Rumor to show that both will persist despite having relatively fewer spreaders.

**Varying the effectiveness of truth spreaders to ignorant people.** For the exposed-spreader interaction model, we investigated varying the effectiveness of Spreaders of Truth to spread the truth to Ignorant people ( $\alpha_+$ ) and varying the effectiveness of Spreaders of Truth to spread the truth to exposed to rumors ( $\epsilon_-$ ). For this, we start with 10 Spreaders of Truth, 10 Spreaders of Rumor and 980 Ignorant people.

To see the effects of varying the effectiveness of spreaders to spread the truth to ignorant people ( $\alpha_+$ ), we let the value of  $\alpha_+$  take on values from [0.2, 0.8] with increments of 0.2 while the other parameters take on values for *Varying  $\alpha_+$*  in Appendix II Table 4.

Appendix III Figure 3 shows that increasing the effectiveness of spreaders to educate ignorant people with the truth ( $\alpha_+$ ) promotes the spread of truth and controls the spread of rumors in a manner similar to what happens in the spreader-spreader interaction model. Appendix III Figure 3(a) shows the proportion of ignorant people in the population decreasing as the effectiveness of Spreaders of Truth to spread the truth increases. The proportion of removed stays relatively the same as shown in Appendix III Figure 3(b). There is a minimal increase in the proportion of removed as more days pass.

Appendix III Figures 3(c) to 3(f) show the truth failing to spread and die out while rumors continually spread and persist. This happens when the Spreaders of Rumor are more effective at spreading rumors ( $\alpha_+ = 0.2 < 0.3 = \alpha_-$ ). The opposite also happens. Truth continually spreads and persists, and rumors fail to spread and die out when the Spreaders of Truth are better than the Spreaders of Rumors in terms of the effectiveness of spreading ( $\alpha_+ = 0.4, 0.6, 0.8 > 0.3 = \alpha_-$ ). But there is a diminishing increase (Appendix III Figures 3(c) and 3(e)) and decrease [Appendix III Figures 3(d) and 3(f)] when jumping from 0.4 to 0.6 to 0.8. This supports the finding from Appendix III Figure 1 that increasing  $\alpha_+$  slightly above  $\alpha_-$  is enough to allow truth to persist and rumors to die out.

The difference here is that the Spreaders of Rumor are spreading the rumors among the exposed to the truth. This leads to a slower reaction to the effect of increasing the effectiveness of the Spreaders of Truth to spread the truth among Ignorant people. Nonetheless, increasing the effectiveness of the Spreaders of Truth to spread the truth among Ignorant people is still important. Doing so allows the truth to persist and rumors to die out in both the spread-spreader and exposed-spreader interaction models.

**Varying the effectiveness of truth spreaders to people exposed to rumors.** To see the effects of varying effectiveness of spreaders to spread the truth to the exposed to rumors ( $\epsilon_-$ ), we let the value of  $\epsilon_-$  take on values from [0.2, 0.8] with increments of 0.2 while the other parameters take on values for *Varying  $\epsilon_-$*  in Appendix II Table 4.

Appendix III Figure 4 shows that increasing the effectiveness of Spreaders of Truth to spread the truth among the exposed to rumors ( $\epsilon_-$ ) would give an immediate impact and the spread of rumors is almost immediately controlled.

Looking at Appendix III Figure 4(a), the proportion of Ignorant people remains the same despite any increase or decrease which is the same when increasing the effectiveness of Spreaders of Truth to stop Spreaders of Rumors from spreading rumors. The proportion of removed also stays the same all throughout no matter how much you increase or decrease the effectiveness of Spreaders of Truth to spread the truth among the exposed to rumors [Appendix III Figure 4(b)].

In Appendix III Figures 4(c) to 4(f), the truth fails to spread and dies out while rumors continuously spread and persist when the Spreaders of Rumor are more effective at spreading rumors to the Exposed to Truth than the Spreaders of Truth at spreading truth to the exposed to rumors ( $\epsilon_- = 0.2 < 0.3 = \epsilon_+$ ). The truth continuously spread and persists while rumors fail to spread and die out when the Spreaders of Truth are more effective at spreading rumors to the exposed to truth than the Spreaders of Rumors at spreading rumors to the exposed to truth ( $\epsilon_- = 0.4, 0.6, 0.8 > 0.3 = \epsilon_+$ ). Though, we can find that the increase in the proportion of exposed to and Spreaders of Truth [Appendix III Figures 4(c) and 4(e)] and the decrease in the proportion of exposed to and Spreaders of Rumors [Appendix III Figures 4(d) and 4(f)] are not proportional when we increase the effectiveness of the Spreaders of Truth to spread the truth among the exposed to rumors from 0.4 to 0.6 to 0.8. This further supports the finding in Appendix III Figure 1, Appendix III Figure 2, and Appendix III Figure 3. It is only required to have the Spreaders of Truth to be slightly more effective than the Spreaders of Rumor for the truth to persist and rumors to die out.

There is a considerable increase in the Spreaders of Truth at the start of the truth and rumor diffusion as compared to Appendix III Figure 1, Appendix III Figure 2, and Appendix III Figure 3 when we increase the effectiveness of the Spreaders of Truth to spread the truth among the exposed to rumors. This further solidifies the need for Spreaders of Truth to take advantage during the first few days of the information diffusion process.

## DISCUSSION

We have analyzed the dissemination of truth and rumor and their interactions with each other using a differential equations approach, and we found the effects of changing the spreading rates on the population. Both spreader-spreader and exposed-spreader interactions have been investigated, which differentiates this from previous work. For each model, we found four equilibrium points, three of which are stable and one unstable. Then, we studied the effects

of varying the spreading parameters on the prevalence of individuals exposed to the truth and exposed to rumor. From our investigation, we have found the following.

- (1) From Appendix III Figures 1(c) and 1(d), we can see that for the proportion of Exposed to Truth to be greater than the proportion of Exposed to Rumor, it is enough that the spreading rate for truth ( $\alpha_+$ ) be greater than the spreading rate for rumors ( $\alpha_-$ ). Similarly, a larger spreading rate for truth results in more Spreaders of Truth than Spreaders of Rumor.
- (2) Increasing  $\alpha_+$  above 0.4 results in diminishing returns for the proportion of Exposed to Truth in the population. When we set  $\alpha_+$  to 0.4 to 0.8, the proportion of exposed remains almost the same, as seen in Appendix III Figure 1(c). The same is true for Spreaders of Truth. However, for Spreaders of Truth, as seen in Appendix III Figure 1(e) the early stages of the simulation show different proportions of Spreaders of Truth, with larger values of  $\alpha_+$  yielding larger proportions of Spreaders of Truth. This indicates an increased effort in spreading the truth.
- (3) From Appendix III Figure 2, we can see that the removal rate for Spreaders of Rumor ( $\delta_-$ ) have little effect on the proportion of exposed individuals. We see from Appendix III Figures 2(c) and 2(d) that it only matters that the removal rate of Spreaders of Truth ( $\delta_-$ ) be greater than the removal rate for Spreaders of Truth ( $\delta_+$ ) for the corresponding type of exposed to prevail over the other ( $E_+ > E_-$ ).
- (4) As seen in Figures 4 and 5, increasing the transmission rate for truth ( $\alpha_+$ ) is much more effective than increasing the removal rate of Spreaders of Rumor ( $\delta_-$ ) in the spreader-spreader model with regards to spreading the truth. This leads to a decrease in the proportion of Ignorant individuals as they become exposed to the truth and, consequently, increase the proportion of Spreaders of Truth. This conclusion is significant to those who are interested in spreading truth to Ignorant individuals. An increase in the activity and convincing power of the Spreaders of Truth should increase the proportion of the population that is exposed to the truth.
- (5) In the exposed-spreader model, increasing the transmission rate of truth is less effective than increasing the conversion rate from Exposed to Truth to Exposed to Rumor in the short term [Appendix III Figures 3(c) and 3(d), and 4(c) and 4(d)]. Similarly, decreasing  $\alpha_+$  decreases the proportion of Ignorant individuals and increases the proportion of Exposed and Spreaders of Truth. This increase is mostly seen in the latter part of the spreading. In addition, there is an immediate increase in the proportion of spreaders in the earlier days of the dissemination process.
- (6) Increasing the conversion rate of those exposed to the truth to exposed to rumor ( $\epsilon_-$ ) has an immediate effect on the spread of rumors. This results in a decrease in the proportion of exposed to rumors, as seen in Appendix III Figure 4(d). Further, there is also a decrease in the proportion of Spreaders of Rumors, as seen in Appendix III Figure 4(f). This also leads to an increase in the proportion of Spreaders of Truth in the beginning [Appendix III Figure 4(e)]. This conclusion would be beneficial to those who want to stop the spread of rumors, which may do so by increasing the effectiveness of Spreaders of Truth in converting individuals Exposed to Rumor to Exposed to Truth.
- (7) Increasing the parameter for the spread of truth ( $\alpha_+$ ) so that it is greater than the parameter for the spread of rumor ( $\alpha_-$ ) leads to an increase in the proportion of exposed and Spreaders of Truth and decrease in proportion for exposed and Spreaders of Rumors. Further, it leads to the eventual disappearance of rumors within the community. However, increasing  $\alpha_+$  to any more than 0.4 would only bring minimal gain. This conclusion is beneficial to those who want to optimize the spread of truth. Spreaders of Truth should aim to be better at transmitting and convincing than Spreaders of Rumor to prevent the spread of rumor, but do not need to be much greater.
- (8) The reproduction numbers generated in the exposed-spreader interaction model are relatively larger than that of the spreader-spreader. This suggests that it is harder to control the epidemic condition in the exposed-spreader interaction model making the spreading of information faster and more efficient.

## CONCLUSION

In this paper, we created two models for the spread of truth and rumor. These models are based on the epidemiological SEIR model, which is a system of coupled ordinary differential equations. We presented a model based on the spreader-spreader interaction, where the presence of Spreaders of Truth affects the presence of Spreaders of Rumor directly. We also presented the exposed-spreader model, in which individuals Exposed to Truth may be influenced by Spreaders of Rumor to switch to being Exposed to Rumor. In our analysis, we showed the truth-rumor-free, truth-free, and rumor-free equilibrium points (Theorems 1– 6), while the truth-rumor coexistence equilibrium point is unstable (Theorem 7). We also showed the stability for the exposed-spreader model in Theorems 8–13. We then ran simulations of both models for different values of spreading parameters  $\alpha_+$ , as well as for different values of the removal rate  $\delta_-$  in the spreader-spreader model, and for different values of the conversion rate  $\epsilon_-$  in the exposed-spreader model. Our results show that the spreading parameter plays a big role in the spread of truth and rumor, in that a larger truth spreading parameter ( $\alpha_+$ ) increases the spread of truth, while also decreases the spread of rumor. We also showed that  $\alpha_+$  only needs to increase to 0.4 in order to have more individuals exposed to truth. Increasing it further does not yield more individuals exposed to truth. The removal parameters also play an important role in the spread of truth and rumor. In the spreader-spreader model, increasing the removal rate of Spreaders of Rumor ( $\delta_-$ ) also decreases the prevalence of rumor in the population. In the exposed-spreader model, we showed that increasing the conversion rate of Exposed to Rumor to Exposed to Truth ( $\epsilon_-$ ) significantly increases the amount of Exposed to Truth in the population, thereby making it an effective mechanism for promoting the spread of truth. Finally, we have shown that in the exposed-spreader model, the spreaders should take advantage of the first few days of the information diffusion process to maximize their influence. To conclude, we have shown that it is harder to control the information epidemic in the exposed-spreader model but information spreads faster in the exposed-spreader model compared to the spreader-spreader model.

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APPENDIX I: PROOFS

**Proposition 1.** The solutions of the system of equations in Equation 1 with positive initial values will remain positive for time  $t$  and are bounded in  $\mathbb{R}_{+0}$ .

**Proof.** Under the given positive initial values, we prove that the components  $I(t), E_+(t), E_-(t), S_+(t), S_-(t)$ , and  $R(t)$  are positive for  $t > 0$  by contradiction.

We assume that there exists a first time  $t_0$  such that  $I(t_0) = 0, I'(t_0) \leq 0$  and  $I(t) > 0$  for  $0 \leq t \leq t_0$ .

Following the assumptions for component I of Equation 1 and after evaluation, we have:

$$I'(t_0) = \Lambda - \alpha_+ I(t_0) S_+(t_0) - \alpha_- I(t_0) S_-(t_0) - \mu I(t_0) = \Lambda > 0.$$

This is a contradiction.

For  $I'(t_0) = \Lambda > 0$  where  $t \in (t - \varepsilon, t_0)$ , this implies that at this interval, we have  $I(t) < 0$  for sufficiently small  $\varepsilon > 0$ , which is also a contradiction. Thus, there exists no such first time  $t_0$  and therefore  $I(t) > 0$  to all  $t \geq 0$ .

Similarly, it can be easily shown that  $E_+(t), E_-(t), S_+(t), S_-(t), R(t) > 0$  to all  $t \geq 0$ .

Next, we show that Equation 1 is also bounded for  $t \geq 0$ . Take the first five equations of Equation 1, we obtain:

$$\begin{aligned} & \frac{d(I + E_+ + E_- + S_+ + S_-)}{dt} \\ &= \Lambda - (I + E_+ + E_- + S_+ + S_-)\mu - (\delta_+ + \delta_-)S_+S_- - \gamma_+S_+ - \gamma_-S_- \\ &\leq \Lambda - (I + E_+ + E_- + S_+ + S_-)\mu \end{aligned}$$

implies that:

$$(I.1) \quad \limsup_{t \rightarrow \infty} (I + E_+ + E_- + S_+ + S_-) \leq \frac{\Lambda}{\mu}$$

For component  $I(t)$ ,

$$\frac{dI}{dt} = \Lambda - \alpha_+ I S_+ - \alpha_- I S_- - \mu I \leq \Lambda - \mu I$$

implies that:

$$I(t) \leq \frac{\Lambda}{\mu} \quad (I.2)$$

For component  $E_+(t)$ :

$$\begin{aligned} \frac{dE_+}{dt} &= \alpha_+ I S_+ - (\beta_+ + \mu) E_+ \\ &\leq \alpha_+ \left(\frac{\Lambda}{\mu}\right) S_+ - \beta_+ E_+ - \mu E_+ \quad , \quad \text{by Equation I.2} \\ &\leq \alpha_+ \left(\frac{\Lambda}{\mu}\right) \left(\frac{\Lambda}{\mu} - E_+\right) - \beta_+ E_+ - \mu E_+ \quad , \quad \text{by Equation I.1} \end{aligned}$$

implies that :

$$E_+(t) \leq \frac{\alpha_+ \Lambda^2}{\mu^2(\alpha_+ \Lambda \mu + \mu^2(\beta_+ + \mu))}$$

Similarly, it can be shown that  $E_-(t), S_+(t), S_-(t)$ , and  $R(t)$  will remain bounded in  $\mathbb{R}_{+0}$ .

**Proposition 2.** The solutions of the system of equations in Equation 2 with positive initial values will remain positive for time  $t$  and are bounded in  $\mathbb{R}_{+0}$ .

The proof is similar to the proof shown in Proposition 1.

**Theorem 12.** The truth-free equilibrium  $E_{ES}^T$  is locally asymptotically stable whenever  $R_{ES}^{01} < \frac{\alpha_+(\beta_- + \mu)}{\alpha_+(\beta_- + \mu) + \mu\epsilon_-}$  and  $R_{ES}^{02} > 1$ .

**Proof.** Take the Jacobian matrix at  $E_{ES}^T$ :

$$\begin{pmatrix} -\mu(R_{ES}^{02} - 1) - \mu & 0 & 0 & -\frac{\alpha_+ \Lambda}{\mu R_{ES}^{02}} & -\frac{\alpha_- \Lambda}{\mu R_{ES}^{02}} \\ 0 & -\beta_+ - \frac{\epsilon_+ \mu}{\alpha_-}(R_{ES}^{02} - 1) - \mu & 0 & \frac{\alpha_+ \Lambda}{\mu R_{ES}^{02}} + \frac{\epsilon_-}{\beta_- + \mu}(R_{ES}^{02} - 1) & 0 \\ \mu(R_{ES}^{02} - 1) & \frac{\epsilon_+ \mu}{\alpha_-}(R_{ES}^{02} - 1) & -\beta_- - \mu & -\frac{\epsilon_-}{\beta_- + \mu}(R_{ES}^{02} - 1) & \frac{\alpha_- \Lambda}{\mu R_{ES}^{02}} \\ 0 & \beta_+ & 0 & -\gamma_+ - \mu & 0 \\ 0 & 0 & \beta_- & 0 & -\gamma_- - \mu \end{pmatrix}$$

whose characteristic polynomial is:

$$X(\lambda) = X_1(\lambda)X_2(\lambda)$$

where:

$$X_1(\lambda) = \lambda^3 + f_1\lambda^2 + f_2\lambda + f_3 \quad (I.3)$$

$$X_2(\lambda) = \lambda^2 + g_1\lambda + g_2 \quad (I.4)$$

Note that  $R_{ES}^{02} > 1$ . In (I.3):

$$f_1 = \mu R_{ES}^{02} + \beta_- + \gamma_- + 2\mu > 0$$

$$f_2 = \frac{\alpha_- \beta_- \Lambda}{\gamma_- + \mu} + \frac{\alpha_- \beta_- \Lambda}{\beta_- + \mu}$$

$$f_3 = \alpha_- \beta_- \Lambda (1 - (R_{ES}^{02})^{-1}) > 0$$

Similar to the proof of Theorem 3, we have:

$$f_1 f_2 - f_3 > 0$$

In (I.4):

$$g_1 = \beta_+ + \gamma_+ + 2\mu + \frac{\epsilon_+\mu}{\alpha_-}(R_{ES}^{02} - 1) > 0$$

$$g_2 = \left(\beta_+ + \frac{\epsilon_+\mu}{\alpha_-}(R_{ES}^{02} - 1) + \mu\right)(\gamma_+ + \mu) - \frac{\alpha_+\Lambda}{\mu R_{ES}^{02}} - \frac{\epsilon_-\Lambda}{\beta_- + \mu}(R_{ES}^{02} - 1)$$

Similar to the proof of Theorem 3, we have:

$$\left(\beta_+ + \frac{\epsilon_+\mu}{\alpha_-}(R_{ES}^{02} - 1) + \mu\right)(\gamma_+ + \mu) > (\beta_+ + \mu)(\gamma_+ + \mu) \quad (I.5)$$

$$-\frac{\alpha_+\beta_+\Lambda}{\mu}(R_{ES}^{02})^{-1} > -\frac{\alpha_+\beta_+\Lambda}{\mu} \quad (I.6)$$

Since  $R_{ES}^{02} > 1$ , then:

$$1 - (R_{ES}^{02})^{-1} > 0 \Rightarrow \frac{\epsilon_-\beta_+\Lambda}{\beta_- + \mu} > \frac{\epsilon_-\beta_+\Lambda}{\beta_- + \mu}(1 - (R_{ES}^{02})^{-1})$$

(I.7)

$$\Rightarrow -\frac{\epsilon_-\beta_+\Lambda}{\beta_- + \mu}(1 - (R_{ES}^{02})^{-1}) > -\frac{\epsilon_-\beta_+\Lambda}{\beta_- + \mu}$$

Since  $R_{ES}^{01} < \frac{\alpha_+(\beta_- + \mu)}{\alpha_+(\beta_- + \mu) + \mu\epsilon_-}$ , then:

$$R_{ES}^{01} < \frac{1}{1 + \frac{\mu\epsilon_-}{\alpha_+(\beta_- + \mu)}} \Rightarrow 1 - R_{ES}^{01} \left(1 + \frac{\mu\epsilon_-}{\alpha_+(\beta_- + \mu)}\right) > 0 \quad (I.8)$$

$$\Rightarrow 1 - \frac{\alpha_+\beta_+\Lambda}{\mu(\beta_+ + \mu)(\gamma_+ + \mu)} - \frac{\epsilon_-\beta_+\Lambda}{(\beta_- + \mu)(\beta_+ + \mu)(\gamma_+ + \mu)} > 0$$

$$\Rightarrow (\beta_+ + \mu)(\gamma_+ + \mu) - \frac{\alpha_+\beta_+\Lambda}{\mu} - \frac{\epsilon_-\beta_+\Lambda}{(\beta_- + \mu)} > 0.$$

Combining (I.5), (I.6), (I.7), and (I.8), we obtain:

$$g_2 > (\beta_+ + \mu)(\gamma_+ + \mu) - \frac{\alpha_+\beta_+\Lambda}{\mu} - \frac{\epsilon_-\beta_+\Lambda}{(\beta_- + \mu)} > 0$$

By Routh-Hurwitz criterion, the roots of  $X_1(\lambda)$  and  $X_2(\lambda)$  are of negative real parts. Therefore,  $E_{ES}^T$  is locally asymptotically stable whenever  $R_{ES}^{01} < \frac{\alpha_+(\beta_- + \mu)}{\alpha_+(\beta_- + \mu) + \mu\epsilon_-}$  and  $R_{ES}^{02} > 1$ .

**Theorem 13.** The rumor-free equilibrium  $E_{ES}^R$  is locally asymptotically stable whenever  $R_{ES}^{01} > 1$  and  $R_{ES}^{02} < \frac{\alpha_-(\beta_+ + \mu)}{\alpha_-(\beta_+ + \mu) + \mu\epsilon_+}$ .

The proof is similar to the proof shown in Theorem 12.

APPENDIX II: TABLES

Table 1. Parameters used for the simulations on the spreader-spreader interaction model (Figure 3).

Scenario	$\Lambda$	$\mu$	$\alpha_+$	$\alpha_-$	$\beta_+$	$\beta_-$	$\gamma_+$	$\gamma_-$	$\delta_+$	$\delta_-$
Truth-Rumor-Free	1	0.3	0.2	0.2	0.7	0.7	0.3	0.3	0.38	0.1
Rumor-Free	2	0.25	0.6	0.1	0.6	0.3	0.15	0.4	0.38	0.1
Truth-Free	2	0.25	0.1	0.6	0.3	0.6	0.4	0.15	0.38	0.1
Coexistence	4	0.3	0.35	0.35	0.65	0.65	0.15	0.15	0.1	0.1

Table 2. Parameters used for the simulate the effects of verifying the effectiveness of truth spreaders on the spreader-spreader interaction model.

Scenario	$\Lambda$	$\mu$	$\alpha_+$	$\alpha_-$	$\beta_+$	$\beta_-$	$\gamma_+$	$\gamma_-$	$\delta_+$	$\delta_-$
Varying $\alpha_+$ (Fig 4)	4	0.3	[0.2, 0.8]	0.35	0.65	0.65	0.15	0.15	0.3	0.3
Varying $\delta_-$ (Fig 5)	4	0.3	0.35	0.35	0.65	0.65	0.15	0.15	0.3	[0.2, 0.8]

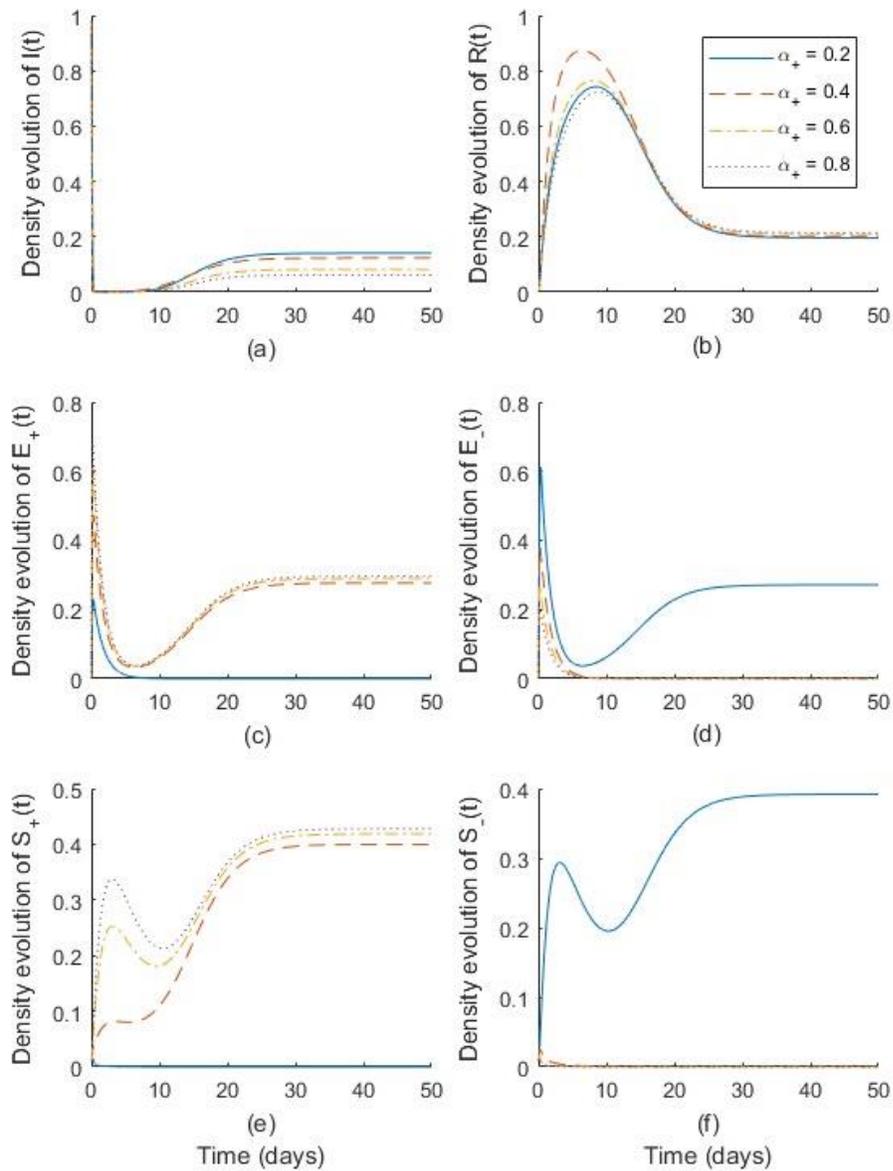
Table 3. Parameters used for the simulations on the exposed-spreader interaction model (Figure 6).

Scenario	$\Lambda$	$\mu$	$\alpha_+$	$\alpha_-$	$\beta_+$	$\beta_-$	$\gamma_+$	$\gamma_-$	$\epsilon_+$	$\epsilon_-$
Truth-Rumor-Free	1	0.3	0.2	0.2	0.7	0.7	0.3	0.3	0.38	0.38
Rumor-Free	2	0.25	0.65	0.1	0.6	0.2	0.1	0.3	0.2	0.38
Truth-Free	2	0.25	0.1	0.65	0.3	0.6	0.3	0.1	0.38	0.2
Coexistence	4	0.3	0.35	0.35	0.65	0.65	0.15	0.15	0.1	0.1

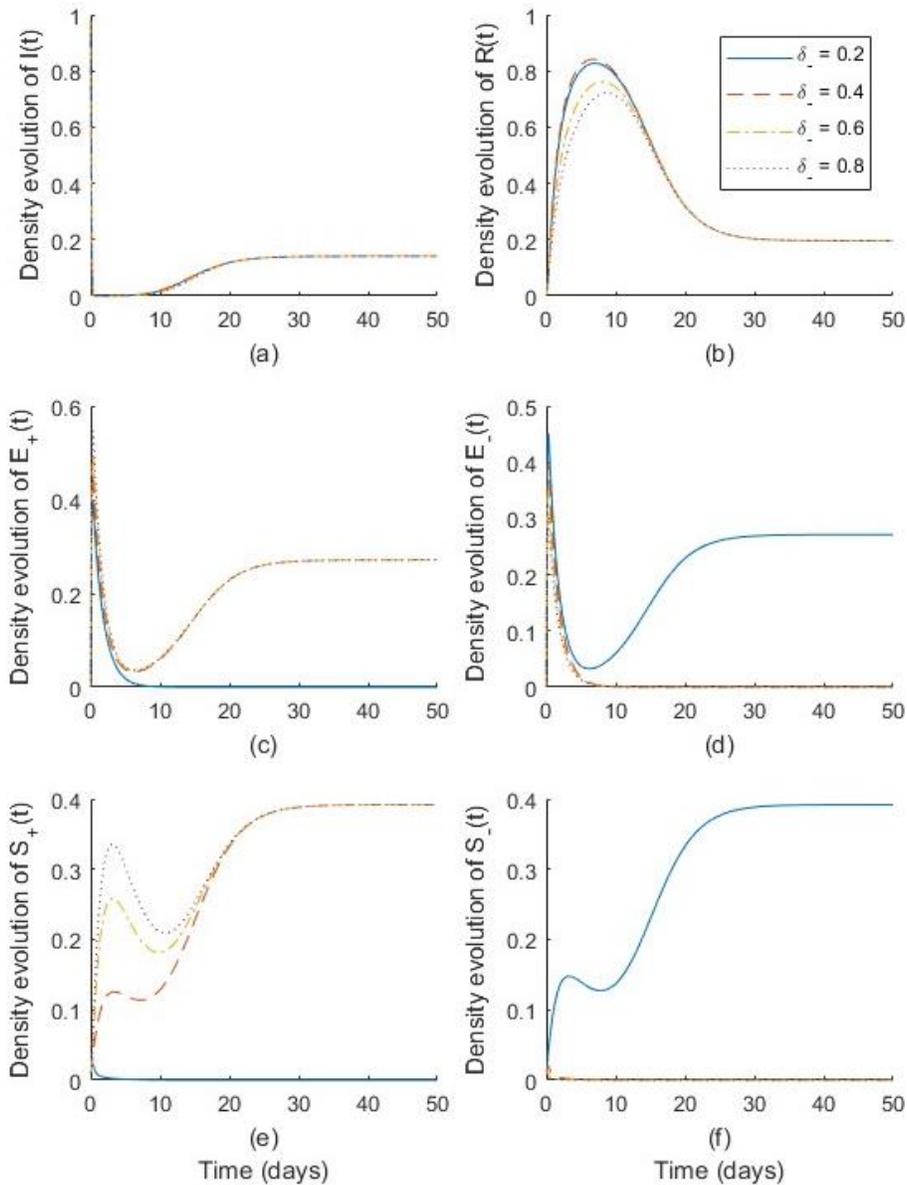
Table 4. Parameters used for the simulate the effects of verifying the effectiveness of truth spreaders on the exposed-spreader interaction model.

Scenario	$\Lambda$	$\mu$	$\alpha_+$	$\alpha_-$	$\beta_+$	$\beta_-$	$\gamma_+$	$\gamma_-$	$\epsilon_+$	$\epsilon_-$
Varying $\alpha_+$ (Fig 7)	4	0.3	[0.2, 0.8]	0.35	0.65	0.65	0.15	0.15	0.3	0.3
Varying $\epsilon_-$ (Fig 8)	4	0.3	0.35	0.35	0.65	0.65	0.15	0.15	0.3	[0.2, 0.8]

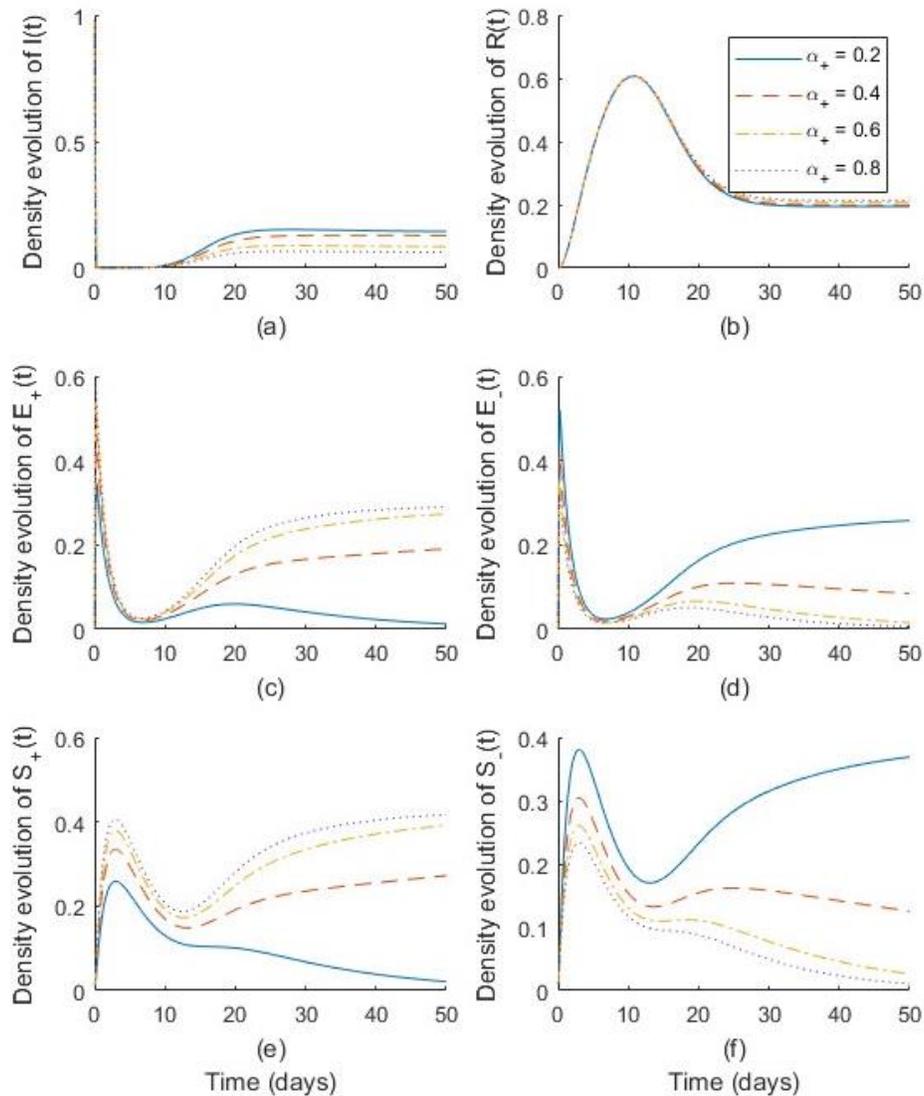
APPENDIX III: FIGURES



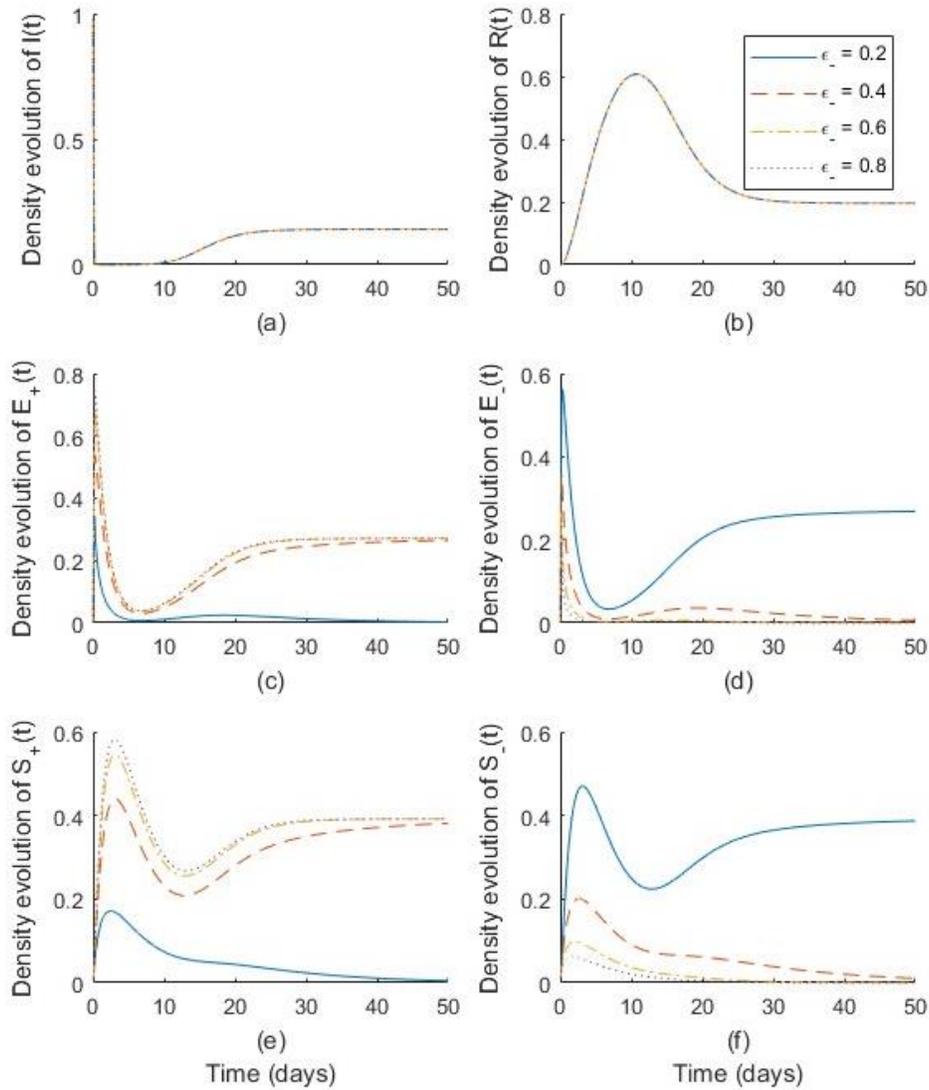
**Figure 1.** Numerical simulations that depict the effects of changing the parameter value  $\alpha_+$  (effectiveness of spreaders to educate Ignorant people with the truth). The truth fails to spread when  $\alpha_+ = 0.2 < \alpha_-$  (c & e). At the same time, the rumor spreads and persists (d and f). However, truth spreads (c and e) when  $\alpha_+ > \alpha_-$  and rumors die out (d and f). Note that it is enough to increase  $\alpha_+$  slightly above  $\alpha_-$  for the rumor to die out. Further increase in  $\alpha_+$  will give diminishing returns.



**Figure 2.** Numerical simulations that depict the effects of changing the parameter value  $\delta_-$  (effectiveness of Spreaders of Truth to convince Spreaders of Rumors to stop spreading). The truth fails to spread when  $\delta_- = 0.2 < \delta_+$  (c and e). At the same time, the rumor spreads and persists (d and f). However, truth spreads (c and e) when  $\delta_- > \delta_+$  and rumors die out (d and f). Note that it is enough to increase  $\delta_-$  slightly above  $\delta_+$  for the rumor to die out. Further increase in  $\delta_-$  will give diminishing returns. Larger values for  $\delta_-$  decreases the proportion of removed (b).



**Figure 3.** Numerical simulations that show the effects of changing the parameter value  $\alpha_+$  (effectiveness of spreaders to educate Ignorant people with the truth) in the exposed-spreader model. Similar to the spreader-spreader interaction, the truth dies out when  $0.2 < \alpha_+$  (c and e) and rumors spread (d and f); truth spreads (c and e) when  $\alpha_+ > \alpha_-$  and rumors die out (d and f). Also, having  $\alpha_+$  slightly above  $\alpha_-$  is enough for rumors to disappear. Further increase in  $\alpha_+$  will give diminishing returns. The difference here is that it takes longer in this model for the truth to spread and rumors to die out and *vice versa*. However, the final proportion for truth is much larger as  $\alpha_+$  increases.



**Figure 4.** Numerical simulations that depict the effects of changing the parameter value  $\epsilon_-$  (effectiveness of spreaders to spread the truth to the exposed to rumors). Similar to the previous simulations, the truth fails to spread (c and e) and rumor prevails when  $\epsilon_- < \epsilon_+$  (d and f). The opposite happens when  $\epsilon_- > \epsilon_+$  (c, d, e, and f). However, the effect on the spread of rumors is much more immediate and effective. It appears that the proportion for those exposed to rumors drops immediately (d). At the same time, the drop is almost as severe for the proportion of Spreaders of Rumor (f)